Credit Risk

Condensed Notes

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Credit risk introduction

Along with market risk and operational risk, credit risk is one of the main risks for banks and companies.

> Credit risk is the risk that the value of a portfolio varies, because of the unexpected changes in the credit quality of counterparty. This risk includes the *default risk* as well as the *credit deterioration risk*, which is the risk that the quality of the borrowing party changes.

Recovery rate

> The recovery rate RR represents the ratio of recovered capital over the face value of the credit, in case of a default (bankruptcy, foreclosure, $\ldots$).

Bank regulations and international standards

Bank regulations

Bank regulations are a form of government regulations which applies to bank. The regulations can include requirements, restrictions, guidelines, supervision, $\ldots$ They aim to reduce different risks of banks. Bank regulations often follow international standards.

International standards for banks

To improve the financial stability of banks, international standards such as the Basel accords have been developed and revised. For credit risk, they aim to set minimum "capital requirements" for banks in order to prevent insolvency in cases of crisis. Credit risk is modeled in order to determine the minimum ratio of capital to total assets (i.e. capital requirements).

> Capital requirements for banks is the amount of liquid capital that banks are required to hold to avoid bad times (crisis, recession, $\ldots$).

From the Basel accords, there are three approaches to the modeling of capital requirements:

- STA: Standardized approach
- F-IRB: Fundamental Internal Rating Based approach
- A-IRB: Advanced Internal Rating Based approach

The computation of the capital requirements depends on the quantity called "Risk-weighted-assets", or RWA. Capital requirements, CR, are typically 8% of RWA.

$$CR = 0.08 \times RWA$$

> Risk-weighted assets is a quantity used to determine capital requirements. It is used as a way to measure the riskiness of assets that have been invested.

Credit ratings

A credit rating is an evaluation of the credit risk associated with a debtor. It describes the probability of the debtor to be able to repay its debt, or probability of default. In other words, it describes credit quality.

> The probability of default is the probability that a debtor will not be able to repay its debt.

> A credit rating is an appreciation of the probability of default of a debtor.

The probability of a debt being repaid depends on the maturity date of the debt. Hence credit ratings are defined for specific time horizons. A maximum time horizon of one-year is commonly used for “short term debt”, and longer horizons are used for “long term debt”.

Usually, credit ratings consist of a grading scale, lower ratings are called “speculative-grade” and higher ratings “investment-grade”.

Credit rating agency

> A credit rating agency is a company that assigns credit ratings to debtors, e.g. companies or sovereign entities.

Credit ratings from agencies approved by the regulator are used in the STA approach to determine the RWA.

Agencies aim for ratings stability by avoiding rating reversal, that is, ratings only change for long-term reasons.

Agencies do not publish ratings for all debtors, such as small companies. For banks required to estimate the probability of default of their counterparties, they need to establish internal ratings to assess the creditworthiness of their clients.

Rating transition matrix

Credit rating agencies publish transition matrices, which represents the empirical probabilities for a rated entity to keep its rating or have its rating changed after a given time horizon. For example, such a matrix can give the probability for a A-rated company to fall into the B-rating next year.

Here is an example of a transition matrix with fictional ratings and probabilities, the class D representing default:

	A	B	C	D
A	0.90	0.08	0.02	0.00
B	0.07	0.85	0.06	0.02
C	0.01	0.09	0.80	0.10

With such tables, we have an estimate of the probability of default of a rated entity.

Cumulative probability of default matrix

Credit rating agencies also publish matrices which give the estimated cumulative probability of default of rated entities with multiple time-horizons.

Here is an example of such matrix:

Initial rating	1 year	2 years	3 years	4 years
A	0.00	0.01	0.04	0.08
B	0.02	0.05	0.10	0.16
C	0.10	0.20	0.33	0.45

With this example, we can see that the probability that a A-rated company defaults the 3rd year or before is 4%. The probability that a A-rated company defaults the 3rd year (and not before) is 0.04 - 0.01 = 3%.

Credit risk modeling

Risk-weighted Assets

The risk-weighted asset (RWA) is a quantity consisting of a company assets or off-balance sheet exposures weighted according to risk.

In can be used as a measure of risk, e.g. it can be used to compare the riskiness of different companies investments.

RWA consists of:

- on-balance sheet assets weighted by their associated risk weights
- credit equivalent amounts for off-balance sheet items, weighted by their associated risk weights

Off-balance sheet items can include, for example, credit default swaps, which include risk due to counterparties, but do not appear in the balance sheet.

The *credit equivalent* amount is a measure prescribed by the regulator which quantifies risk for off-balance sheet assets.

Standardized approach

The STA approach uses credit ratings from external regulator-approved agencies to weight the asset classes.

This approach is the simplest of the three approaches to compute the RWA, and is therefore favored by smaller-sized banks.

$$RWA = \sum_{i=1}^{N} \alpha_i E_i + \sum_{i=1}^{M} w_i C_i$$

N: the number of on-balance sheet items
α_i: the weight of the i-ath on-balance sheet item
E_i: the principal amount of the i-th on-balance sheet item
M: the number of off-balance sheet items
w_i: the weight of the i-th off-balance sheet item
C_i: the credit equivalent amount of the i-th off-balance sheet item.

In the STA approach, risk weights are defined by the regulator, for classes of items and ordered according to credit ratings.

As an example, here is a fictional table of risk weights, using fictional ratings A, B and C:

Entity	A	B	C	Unrated
Country	0%	50%	150%	100%
Company	20%	70%	150%	100%

With this example, assets invested in credit for a B-rated company will be added to the RWA with a 70% weight.

Fundamental internal rating based approach

In the F-IRB approach, banks have more flexibility in the determination of the RWA. Essentially, they can use internal models to determine the probability of default PD of their counterparties.

Once the PD obtained for all items, banks will use formulas defined by the regulator to determine the RWA.

Credit requirements derivation

Assuming we have obtained the probability of default of all counterparties, we can derive the capital requirements (or RWA) following the F-IRB approach. The derivation is determined by the regulator.

We consider a bank that has multiple (N) debt counterparties. Let, for all i in $[\![1, N]\!]$ PD_i be the 1-year probability of default of counterparty i. Assume a constant correlation ρ between each pair of counterparties default events. Banks often use a one-factor Copula model to estimate the default rate distribution for the group of counterparties, then compute the $WCDR$, or *worst case default rate*. $WCDR$ is defined as the 99.9% quantile of the default rate distribution.

$$WCDR_i = \Phi(\frac{\Phi^{-1}(PD_i) + \sqrt{\rho}\Phi^{-1}(0.999)}{\sqrt{1-\rho}})$$

Then we can estimate the VaR for all counterparties:

$$VaR_{0.999}^{1-year} = \sum_{1 \leq i \leq N} EAD_i \times LGD_i \times WCDR_i$$

And the expected credit loss:

$$EL = \sum_{1 \leq i \leq N} EAD_i \times LGD_i \times PD_i$$

We can compute the correlation parameter for corporate, sovereign and bank exposures with the following equation obtained from the regulator:

$$\rho_i = 0.12(1 + e^{-50 \times PD_i})$$

Then, for corporate, sovereign and bank exposure, the capital requirements for counterparty i is computed as:

$$CR_i = EAD_i \times LGD_i \times (WCDR_i - PD_i) \times MA_i$$

Where MA_i is the maturity-adjustment, which can be computed with PD_i, the maturity M_i and a formula provided by the regulator:

$$MA_i = \frac{1 + (M_i - 2.5)b_i}{1 - 1.5b_i} \quad b_i = (0.11852 - 0.05478 \times log(PD_i))^2$$

We have finally:

$$CR = \sum_{1 \leq i \leq N} EAD_i \times LGD_i \times (WCDR_i - PD_i) \times MA_i$$

For retail exposures, we have:

$$\rho_i = 0.03 + 0.13e^{-35PD_i} \quad CR_i = EAD_i \times LGD_i \times (WCDR_i - PD_i)$$

Note that there is no maturity adjustment term.

Advanced internal rating based approach

In the A-IRB approach, banks can determine the PD of counterparties as in the F-IRB approach, but can also determine other quantities used to determine RWA, in particular the following quantities:

- PD: probability of default
- EAD: exposure at default
- LGD: loss given default

The exposure at default EAD is the worst-case loss for the bank for a given debt. The loss given default LGD is the expected loss for the bank in case of a default, expressed as a percentage over the EAD.

With RR being the recovery rate, we have:

$$LGD = EAD \times (1 - RR)$$

Those parameters are determined using internally developed models that have to be approved by the regulator.

Risk measure

A risk measure is a function that is used to quantify risk.

$$\phi : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$$

With $\mathcal{L}$ is a set of random variables representing a portfolio returns, with outcomes in $\mathbb{R} \cup \{+\infty, -\infty\}$. We require, by abusing the notation, $\mathbb{R} \subset \mathcal{L}$ and $\forall a \in \mathbb{R}\ \phi(a) = -a$.

A risk measure follows certain properties:

- normalized: $\phi(0) = 0$
- translation invariance: $\forall a \in \mathbb{R}, z \in \mathcal{L}\ \phi(z + a) = \phi(z) - a$
- monotonicity: $\forall (z_1, z_2) \in \mathcal{L}^2\ z_1 \leq z_2 \implies \phi(z_1) \geq \phi(z_2)$

Interpretation: - normalized: the risk associated with an empty portfolio is null - translative: adding a non-risky asset to a portfolio decreases its risk by the value of the non-risky asset - monotone: The risk associated with a higher-return portfolio is less than the risk associated with a lower-return portfolio

Coherent risk measure

A measure of risk $\phi : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ is coherent if it follows the following properties:

- sub-additive: $\forall (z_1, z_2) \in \mathcal{L}^2\ \phi(z_1 + z_2) \leq \phi(z_1) + \phi(z_2)$
- positive homogeneous: $\forall a \in \mathbb{R}^+\ \forall z \in \mathcal{L}\ \phi(az) = a\phi(z)$

Interpretation: - sub-additive: diversification reduces risk - positive homogeneous: the risk of a portfolio scales linearly with the portfolio value

Value at risk

The Value at Risk VaR is a measure that is used to assess credit risk. The VaR is called $C-VaR$ or Credit-VaR when talking about credit risk.

$$VaR_\alpha^T = inf\{l \in \mathbb{R} : \mathcal{P}(L > l) \leq 1 - \alpha\} = inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}\ \alpha \in [0, 1]$$

Where L is a random variable representing the losses, and F_L is the cumulative probability distribution function of L. The loss distribution is defined over a defined time horizon T, often implied.

The α value is often determined by law or prescriptions. Common values are 0.95, 0.99, …

The $VaR_{0.999}^{1-year}$ is used for determining the capital requirements for banks.

In general, the value at risk is not a coherent risk measure.

If μ is the mean of the loss distribution, we define the mean value at risk as such:

$$VaR_{\alpha}^{mean} = VaR_{\alpha} - \mu$$

Expected shortfall

The expected shortfall ES is a measure of risk.

$$ES_{\alpha} = E[L|L \geq VaR_{\alpha}]$$

Where L is a random variable representing the losses, and VaR_{α} the value at risk for L.

The expected shortfall is always coherent.

F-IRB models

In the F-IRB and A-IRB approaches, banks can use ratings or default models to estimate the probability of default of a counterparty. Ratings are either internally or externally developed. Models of default are divided in structural models and other models.

In this section are described high-levels examples of default probability estimation. More detailed models are described in later sections.

Rating-based estimation

When using credit ratings to estimate the probability of default, we assume that credit ratings fully determines the PD of a counterparty.

Rating agencies provide historical default probabilities as well as equity-based predictions.

Altman Z-score estimation

Altman Z-score is a prototype of internal-rating methods based on discriminant analysis. It is a *financial distress index*, with a typical time horizon of one year. There exists different versions of Altman's Z-score depending on the type of the debtor (industry, private/public, emerging markets, …) and analyst preference.

$$Z_{score} = 1.2R_1 + 1.4R_2 + 3.3R_3 + 0.6R_4 + 1.0R_5$$

With:

$$R_1 = \frac{\text{Working Capital}}{\text{Total Assets}} \quad R_2 = \frac{\text{Retained Earnings}}{\text{Total Assets}} \quad R_3 = \frac{\text{EBIT}}{\text{Total Assets}} \quad R_4 = \frac{\text{Sales}}{\text{Total Assets}}$$

The Z-score is then compared to a scale:

$$Z_{score} > 2.99 \rightarrow \text{safe zone} \mid 1.81 < Z_{score} < 2.99 \rightarrow \text{grey zone} \mid Z_{score} < 1.81 \rightarrow \text{distress zone}$$

Note that the Z_{score} does not account of off-balance sheet items. This constitutes one of its limitation. Moreover, it only is not forward-looking: it only captures information from the past.

Estimation using structural models

Structural models are models where default occurs when a stochastic variable falls below a given threshold. Structural models are also called *threshold models*.

As an example, we a company asset value can represent the stochastic variable, and the company liabilities the threshold: default occurs when the company assets fall below the company liabilities.

A-IRB models

In this section are described high-levels examples of A-IRB models. More detailed models are described in later sections.

Structural models of default

Merton's model

Merton's model is a prototype of structural models of default. It is a simple model, PD is easily computable, but has limitations.

We consider a limited company that can finance itself with equity or debt, and does not pay dividends before time T.

- Let V_t represent the company asset value at time t.
- Let B represent the principal value of the (only) debt of the company, which is a zero-coupon bond with maturity T.
- Let B_t represent the company debt at time t.
- Let S_t represent the value of equity at time t.

Markets are assumed friction-less, therefore

$$\forall t \in [0, T] \; V_t = S_t + B_t$$

Default happens if and only if $V_T \leq B$. In case of default, the company is liquidated, with $B_T = V_T$ and $S_T = 0$.

In the general case:

$$S_T = (V_T - B)^+ = max(0, V_T - B) B_T = B - (B - V_T)^+ = min(B, V_T)$$

This implies that S_T corresponds to the payoff of a European call option on V_T with a premium of B.

Applying the Black-Scholes-Merton model to value the equity at $t = 0$, we get:

$$S_0 = V_0\Phi(d_1) - Be^{-rT}\Phi(d_2)$$

With:

$$d_1 = \frac{log(V_0/B) + (r + \sigma_V^2/2)T}{\sigma_V\sqrt{T}} \quad d_2 = d_1 - \sigma_V\sqrt{T}$$

Φ is the CDF of a standard Gaussian $\mathcal{N}(0,1)$
σ_V is the volatility of assets (assumed to be constant)
r is the risk-free rate on the market

The probability of default can be shown to be:

$$P(V_T \leq B) = \Phi(-d_2)$$

Therefore, the probability of default depends on parameters V_0 and σ_V, which cannot be observed directly, they need to be inferred.

If we know S_0 (e.g. the company is publicly traded), it can be shown that:

$$S_0 = \frac{\sigma_V}{\sigma_S}\Phi(d_1)V_0$$

Where σ_S is the instantaneous volatility of equity, which can be observed in the market. This formula, together with the Black-Scholes-Merton formula can be used to derive V_0 and σ_V.

Limitations

Here are some of the biggest limitations of the model: - default can only happen at a certain time T - default mandatorily triggers a full liquidation of the company - the use of the Black-Scholes valuation does not consider enough "exceptional" events, i.e. it assumes normality

KMV model

The KMV model tries to improve Merton's model by minimizing some of its limitations. It is proprietary and not completely open, therefore the description below is non complete and may not be correct.

KMV introduces the *expected default frequency*, EDF, which is the probability that a firm will default within one year following the KMV model.

We can compute the EDF following Merton's model with $T = 1$ year, the 1-year PD of a firm is given by $P(V_1 \leq B)$. Hence,

$$EDF_{Merton} = P(V_1 \leq B) = \Phi(\frac{log(B) - log(V_0) - (r - \sigma_V^2/2)}{\sigma_V})$$

With $\forall x \in \mathbb{R}\ \Phi(x) = 1 - \Phi(-x) = \bar{\Phi}(-x)$, we can rewrite the previous formula:

$$EDF_{Merton} = \bar{\Phi}(\frac{log(V_0) - log(B) + (r - \sigma_V^2/2)}{\sigma_V}) = \bar{\Phi}(A)$$

With $A = \frac{log(V_0) - log(B) + (r - \sigma_V^2/2)}{\sigma_V}$.

In the KMV's EDF, $\bar{\Phi}$ is substituted by some decreasing function that is estimated using empirical data, $\bar{F}_{KMV}$, that better represent extreme events.

B is replaced by another threshold $\tilde{B}$.

A is replaced by the simpler expression $\tilde{A} = DD$.

Where DD is the *distance to default* quantity:

$$DD = \frac{V_0 - \tilde{B}}{\sigma_V V_0}$$

Finally, we have:

$$EDF_{KMV} = \bar{F}_{KMV}(\bar{A}) = \bar{F}_{KMV}(\frac{V_0 - \tilde{B}}{\sigma_V V_0})$$

CreditMetrics model

CreditMetrics is a Merton-like model, in which thresholds are not defined by liabilities, but by credit ratings.

Assume a firm F, and a time horizon $[\![0, T]\!]$. Assume that credit ratings are associated with classes $[\![0, n]\!]$, sorted from worst to best, e.g. class 0 represents the case of default.

We define, for $j \in [\![0, n]\!]$, $\bar{p}(j)$ as the probability that F is in class j at time T. Hence $\bar{p}(0)$ is the probability of default at time T. The $\bar{p}(j)$ can be obtained from the transition matrix given by the credit ratings, and the initial ($t = 0$) rating of F.

We define the series $-\infty = \tilde{d}_0 < \tilde{d}_1 < ... < \tilde{d}_n < \tilde{d}_{n+1} = +\infty$ to be the thresholds from classes, that is, F is in class j if and only if $\tilde{d}_j < V_T \leq \tilde{d}_{j+1}$.

We modify the values V_T and $\tilde{d}_j$ using the same standardization:

$$X_T = \frac{log(V_T) - log(V_0) - (r - \sigma_V^2/2)T}{\sigma_V \sqrt(T)};\ \forall j \in [\![0, n]\!]\ d_j^* = \frac{log(\tilde{d}_j) - log(V_0) - (r - \sigma_V^2/2)T}{\sigma_V \sqrt(T)}$$

Therefore, F is in class j if an only if $d_j^* < X_T \leq d_{j+1}^*$.

We can compute the series $(d_j^*)_{j \in [\![0, n+1]\!]}$ as such:

$$\forall j \in [\![1, n]\!]\ d_j^* = \Phi^{-1}(\sum_{i \leq j} \bar{p}(i))$$

Mixture models

In a mixture model, the default probabilities (P_i) are assumed to depend on a set of common economic factors.

CreditRisk+

In CreditRisk+, the default probabilities are assumed to be conditionally Poisson distributed. Here we summarize a simplified model.

We assume a large portfolio of loans, each with a small probability p of default. Since n is large, p is small and we assume default events to be independent, we can estimate the number of defaults X to follow a Poisson distribution of rate $\lambda = np$.

The probability of observing $k \in \mathbb{R}^+$ defaults is:

$$\mathcal{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Using the distribution of number of defaults and the distribution of the losses when defaults (estimated empirically), we can derive the probability distributions of the total losses from defaults.

With the resulting distribution, quantities such as the VaR and ES can be computed.

Stress testing and back testing

Back testing

Back-testing is a validation procedure used in risk management to assess the accuracy of chosen risk measures on historical data. After back-testing, one can comment if the risk measure has been over-estimated or under-estimated.

Back-testing VaR

To back-test a computed value at risk VaR_α, we can count the number of days in the past in which the actual loss was higher than the computed VaR_α. Those "exceptional" days should have happened at a rate less than $1 - \alpha$. Standard back-testing is based on Bernoulli trials generating binomial random variables.

If the VaR_α under test is accurate, the probability p of observing an exception is $1-\alpha$. Looking over historical data over n days, assuming we observe $m < n$ exceptions. We can compare p and m/n, using a 5% confidence level (for example).

$$\mathcal{P}(\text{observing at least m exceptions in n days}) = \mathcal{P}(X \geq m) = 1 - pbinom(m, n, p)$$

Where $X \sim B(n, p)$ is a random variable following the binomial distribution with n trials and probability $p = 1 - \alpha$ of success (hypothesis).

If the computed probability $\mathcal{P}(X \geq m)$ is less than 5% (the significance level), we can reject the hypothesis that p is the exception probability.

Stress testing

Stress tests are procedures to assess the resilience of a financial institution in cases of crisis or unexpected events.

For example, stress tests can be performed in order to check if banks remain solvent if the interest rates increase, or if the default correlation among counterparties increases. The set of settings, or parameters, used in the stress tests is called the scenario. Scenarios are derived through historical events, expert judgment or decisions from the regulator.

With the Basel III accords, stress testing is part of requirements for large financial institutions.

Stress testing is done using statistical tools, e.g. monte-carlo simulations, extreme value theory…

Stressed Value at risk

Stressed-Value-at-Risk ($S - VaR$) corresponds to the value at risk only considering the worst losses, usually largest half of losses. It will be larger than VaR for the same given losses, and can be used as a more conservative risk quantity than VaR.